Motion of a conducting floating object on top of an electrolyte submitted to Lorentz forces

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Context

- Electromagnetic fluid flow control :
electrical current + *magnetic induction* $\Longrightarrow$ Lorentz force density $\Longrightarrow$ *flow*

- Small scale device : milli-, micro- (and even nano-) meters, microfluidics.

- Chaotic advection : ideally $\vec{v}(t, \vec{x})$ such as

$$\forall \epsilon > 0 \ : \ \frac{d\vec{X}}{dt} = \vec{v}(t, \vec{X}) \Longrightarrow \forall \ \vec{x}, \ \exists \ t \ : \ |\vec{X}(t) - \vec{x}| < \epsilon$$

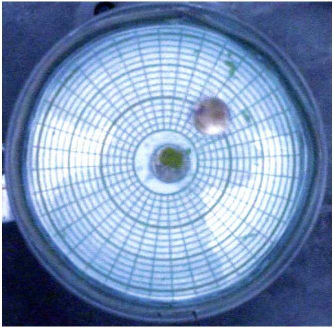
- Chaotic advection enhances mixing in Stokes flows (small scale).

- Lorentz forces are effective in propelling liquids in microfluidics.

Work's rationale

- An object floating on a liquid electrolyte can be used as a stirrer (mixer). If the liquid is propelled by Lorentz forces and if the stirrer can conduct electricity, a very complex flow can thus be created ;
- The floating object has its own motion with respect to the flow ;
- This work is a first step to evaluate the mixing ability of a floating object when both fluid and object are submitted to Lorentz forces ;
- The results obtained with a numerical coupled electromagnetic/fluid flow model of the motion of a conducting floating object on top of an electrolyte are compared with experiments.

Experimental setup



Tank and F.O. (floating object)

zoom on the F.O.
$\varnothing$ 5 mm ; thickness 0.35 mm

- Tank : $\varnothing$ 5 cm ; Electrolyte ($CuSO_4$) : depth 2 mm
- Central ($\varnothing$ 5 mm) and peripheral electrodes : voltage $\approx 0.5\ V$

Steady-state Motion

- When the applied voltage is constant, the steady state trajectory of the F.O. is circular. R is the equilibrium radius.
The free surface of the electrolyte is slightly concave (meniscus) due to surface tension. An equilibrium between the attractive central force and the repulsive centrifugal forces is obtained.

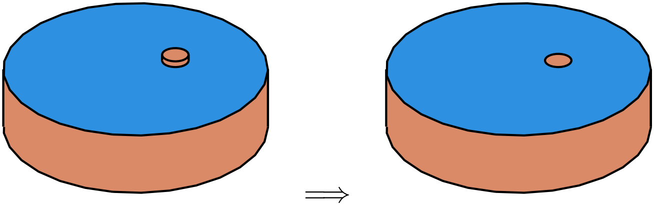
- The position of the center of the F.O. is

$$R\ \vec{k}_R \text{ with } \begin{cases} \vec{k}_R = \cos\Theta\ \vec{k}_x + \sin\Theta\ \vec{k}_y \\ \vec{k}_\Theta = -\sin\Theta\ \vec{k}_x + \cos\Theta\ \vec{k}_y \end{cases}$$

- Furthermore the F.O. has an intrinsic rotation (velocity ω) ;

- The aim here is to determine $\dot{\Theta}$ and ω vs geometry, physical properties, applied voltage and magnetic field intensity.

Electric model



- Two regions : the electrolyte ($\sigma \approx 1.5 S/m$), the thin ($0.35\ mm$) floating object ($\sigma_c \approx 50 MS/m$)

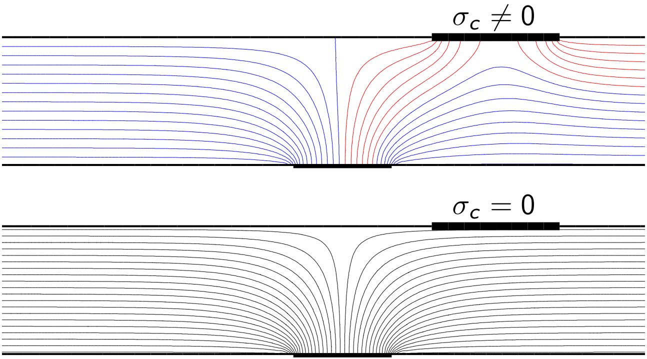
- Asymptotic model : the F.O. is approximated as 2d part of the (assumed flat) free surface

- the electric potential φ minimizes the Joule dissipated power

$$\int_D \sigma \left(\vec{\nabla}\varphi\right)^2 d\vec{x} + \int_\Gamma e\ \sigma_c \left(\vec{\nabla}_{2d}\varphi\right)^2 d\vec{x}_{2d}$$

with given voltages on the electrodes.

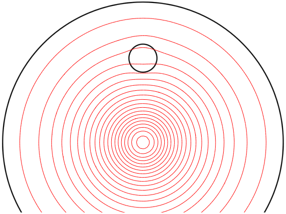
Path of electrical current (in a cross section)



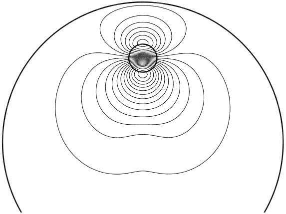
- The conducting F.O. deflects a non-negligible amount of the current : $\approx 10\%$ of the total current here ($R = 6\ mm$).

Lorentz force density in the electrolyte

- $\vec{b}_s$ computed with the Biot and Savart formula, $\vec{f}_l = \vec{j} \times \vec{b}_s$



f_l conducting F.O.



Δf_l between cond. and non-cond. F.O.

- the conducting F.O. generates a dipolar perturbation in the electrolyte's force density.

- Moreover, the force and torque in the F.O. are

$$F_{l\Theta} = \int_{\Gamma} -(\sigma_c \; e \; \vec{\nabla}_{2d}\varphi \times \vec{b}_s) \cdot \vec{k}_{\Theta} \; d\vec{x}_{2d}$$

$$\Gamma_l = \int_{\Gamma} -\left[(\vec{x} - R \; \vec{k}_R) \times (\sigma_c \; e \; \vec{\nabla}_{2d}\varphi \times \vec{b}_s)\right] \cdot \vec{k}_z \; d\vec{x}_{2d}$$

Velocity field

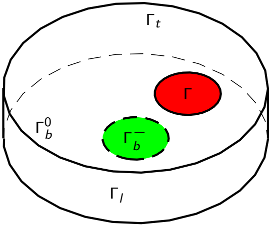
- Stokes approximation (inertial terms neglected) is used ;

$\vec{v}$ the velocity and p the pressure fields

$$\vec{f}_l = -\eta \vec{\nabla}^2 \vec{v} + \vec{\nabla} p \quad ; \quad \vec{\nabla} \cdot \vec{v} = 0$$

balance of local forces — incompressibility

- Boundary conditions



$$\vec{v} = u\ \vec{k}_x + v\ \vec{k}_y + w\ \vec{k}_z$$

- $u = v = w = 0$ on $\Gamma_b \cup \Gamma_l$
- $w = 0$; $\partial_z u = \partial_z v = 0$ on Γ_t
- $w = 0$;

$u\ \vec{k}_x + v\ \vec{k}_y = R\ \dot{\Theta}\ \vec{k}_\Theta + \omega\ \vec{k}_z \times (\vec{x} - R\ \vec{k}_R)$ on Γ

- Velocity split

$\vec{v} = \vec{v}_0 + R\ \dot{\Theta}\ \vec{v}_1 + \omega\ \vec{v}_2$ $(p = p_0 + R\ \dot{\Theta}\ p_1 + \omega\ p_2)$,
which leads to 3 independent Stokes problems.

Stokes functionals

- Common constraints : $u = v = w = 0 \quad \text{on } \Gamma_b \cup \Gamma_l,$
  $w = 0 \,;\; \partial_z u = \partial_z v = 0 \quad \text{on } \Gamma_t$

- Velocity $\vec{v}_0$ and pressure p_0 when the F.O. is at rest

$$D_0 = \int_D \left[\frac{\eta}{2} \left(\vec{\nabla} \times \vec{v} \right)^2 + \vec{\nabla} p \cdot \vec{v} - \vec{f}_l \cdot \vec{v} \right] d\vec{x}$$

with common constraints and $\vec{v} = \vec{0}$ on Γ

- $\vec{v}_1$ and p_1 without Lorentz force density $\vec{f}_l$ nor intrinsic rotation ω

$$D_1 = \int_D \left[\frac{\eta}{2} \left(\vec{\nabla} \times \vec{v} \right)^2 + \vec{\nabla} p \cdot \vec{v} \right] d\vec{x}$$

with common constraints and $\vec{v} = \vec{k}_\Theta$ on Γ

- $\vec{v}_2$ and p_2 without $\vec{f}_l$ nor azimuthal rotation $\dot{\Theta}$

$$D_2 = \int_D \left[\frac{\eta}{2} \left(\vec{\nabla} \times \vec{v} \right)^2 + \vec{\nabla} p \cdot \vec{v} \right] d\vec{x}$$

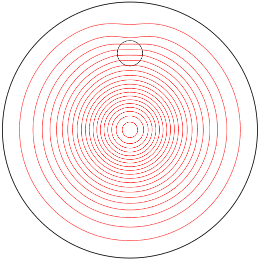
with common constraints and $\vec{v} = \vec{k}_z \times (\vec{x} - R\,\vec{k}_R)$ on Γ

Extremization of the functionals

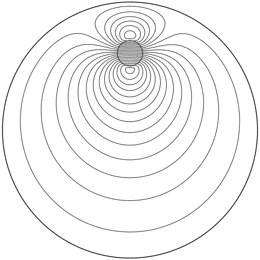
- For a functional, say $D_0(\vec{v}, p) = \int_D \left[\frac{\eta}{2} \left(\vec{\nabla} \times \vec{v} \right)^2 + \vec{\nabla} p \cdot \vec{v} - \vec{f}_l \cdot \vec{v} \right] d\vec{x}$ (taking into account the Dirichlet condition) : formally the extremization consists in a simultaneous :
  - minimization with respect to $\vec{v}$ – D_0 express the dissipated power by viscous effets minus the rate of work of the force density $\vec{f}_l$ by unity of time, i.e. the source power ;
  - maximization with respect to p – the term $\int_D \vec{\nabla} p \cdot \vec{v}\, d\vec{x} = -\int_D p\, \vec{\nabla} \cdot \vec{v}\, d\vec{x}$ ensures the incompressibility constraint and p is the Lagrange multiplier associated to this constraint.

- The Uzawa algorithm is used : $\vec{v}(p)$ minimizes $D_0(\vec{v}, p)$ with respect to $\vec{v}$ when p is given and the functional $D_0(\vec{v}(p), p)$ is mazimized with respect to p (at the discrete level, by using the conjugate gradient method).

Streamlines and iso-velocity lines

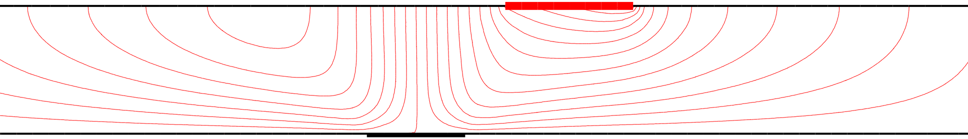


Conducting F.O.

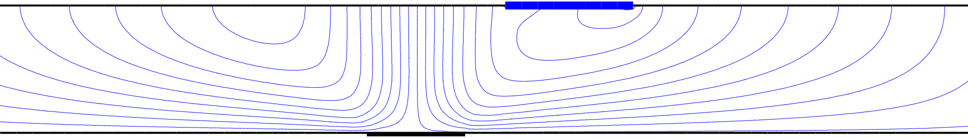


diff. between a cond and non-cond F.O

Cond. F.O.



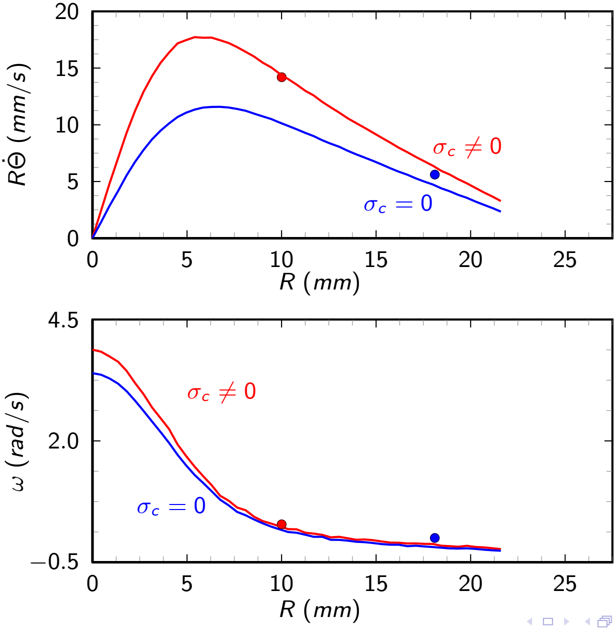
Non-cond. F.O.



Balance of force and torque in the F.O.

- Lorentz : $F_{l\Theta} = \int_\Gamma -(\sigma_c \, e \, \vec{\nabla}_{2d}\varphi \times \vec{b}_s) \cdot \vec{k}_\Theta \, d\vec{x}_{2d}$
$\Gamma_l = \int_\Gamma - \left[(\vec{x} - R \, \vec{k}_R) \times (\sigma_c \, e \, \vec{\nabla}_{2d}\varphi \times \vec{b}_s) \right] \cdot \vec{k}_z \, d\vec{x}_{2d}$
- Friction $F_f = \int_\Gamma \vec{f}_f \cdot \vec{k}_\Theta \, d\vec{x}_{2d}$,
$\Gamma_f = \int_\Gamma \left((\vec{x} - R \, \vec{k}_R) \times \vec{f}_f \right) \cdot \vec{k}_z \, d\vec{x}_{2d}$ with
$\vec{f}_f = \eta \left(\partial_z u \, \vec{k}_x + \partial_z v \, \vec{k}_y \right)$ on Γ (shear stress analysis).
- Steady-state : $F_{l\Theta} + F_f = 0$ and $\Gamma_l + \Gamma_f = 0$
with the splitting $\vec{v} = \vec{v}_0 + R \, \dot{\Theta} \, \vec{v}_1 + \omega \, \vec{v}_2$ *:*
two equations and two unknowns $\dot{\Theta}$, ω

Velocity and intrinsic velocity of the F.O.



Simplified dynamical system

- With the quasi-static assumption for both the electric potential and the velocity field, the motion of the F.O. is given by

$$mR\frac{d\dot{\Theta}}{dt} = F_{I\Theta} + F_f \quad ; \quad J\frac{d\omega}{dt} = \Gamma_I + \Gamma_f$$

where

$$\begin{bmatrix} F_f \\ \Gamma_f \end{bmatrix} = -\begin{bmatrix} \lambda_0 \\ \xi_0 \end{bmatrix} - \begin{bmatrix} \lambda_{1\Theta} & \lambda_{1\omega} \\ \xi_{1\Theta} & \xi_{1\omega} \end{bmatrix} \begin{bmatrix} R\dot{\Theta} \\ \omega \end{bmatrix}$$

- The order of the relaxation time $\approx 1s$ is qualitatively verified by experiment.

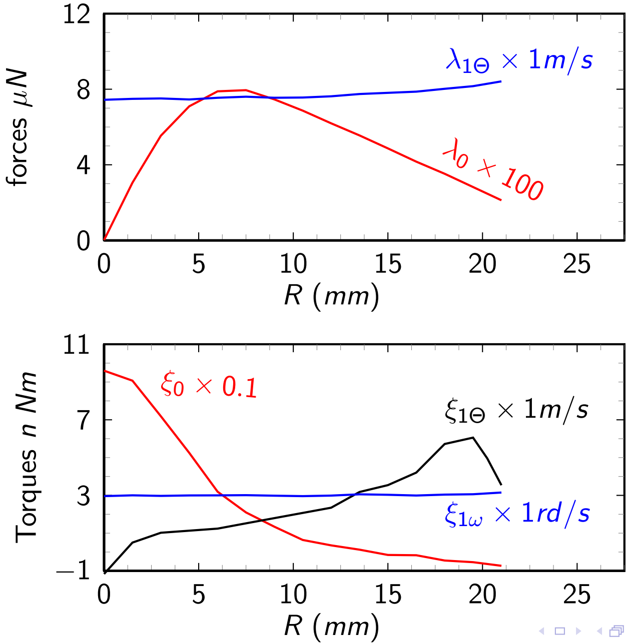
Conclusions

- A coupled electromagnetic/fluid flow model has been constructed and solved to calculate the steady state of the motion of a floating object. Results agree quite well with experiments.

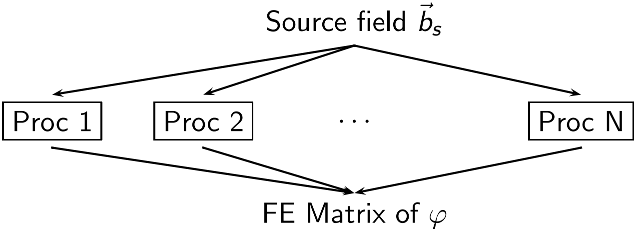
- Future work is underway to predict the motion when the applied voltage is time-dependent and for different geometries.

- The final objective is to design a device where an additive will be mixed with the fluid by chaotic advection.

Friction coefficients vs R



Parallel Computation



parallel solver : MUMPS $\rightsquigarrow$ φ

Source field $\vec{f}_l$

FE Matrices of (u, v, w)

parallel solver : MUMPS

Pressure update $\rightsquigarrow$ (u, v, w, p)