No-slip motion of a spherical magnet on top of a conductive plate

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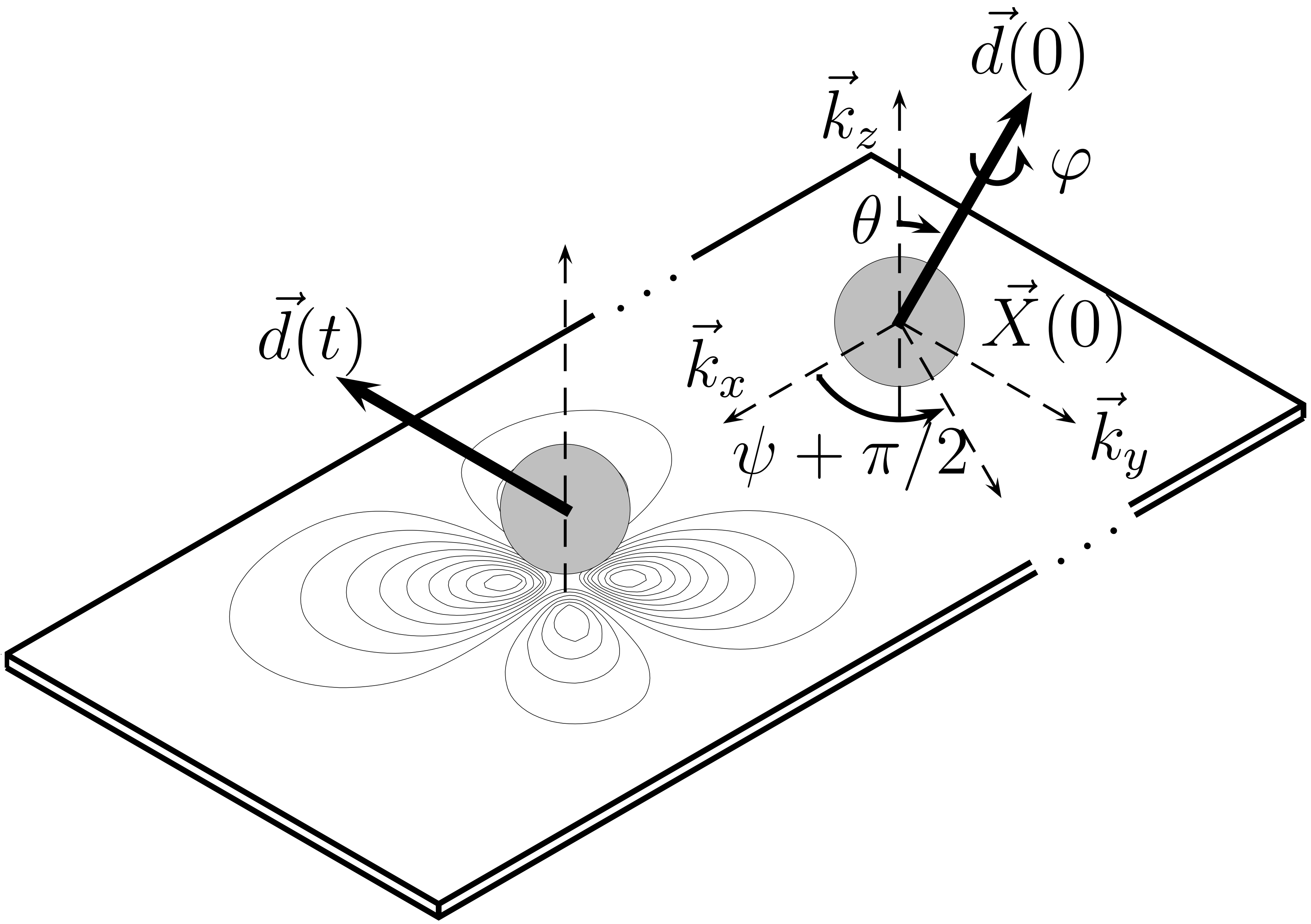
The trajectory of a spherical magnet which rolls without slipping on a conductive plate is modelled. A time-stepping $\vec{T}-\Omega$ method is used to find the electromagnetic force and torque. Due to the free motion of the magnet, the motion is not reducible to a single friction force coefficient, the degrees of freedom involve the position as well as the direction of magnetization. The trajectory is computed and compared to performed experiments.

Nd-Fe-B Magnet on a copper plate



Nd-Fe-B magnet

Radius : $R = 6.35\ mm$

Mag. moment : $\mathcal{M} = 1\ A/m^2$

Copper plate

Thickness : $5\ mm$

Conductivity : $\sigma = 50\ MS/m$

Center : $\vec{X} = X(t)\ \vec{k}_x + Y(t)\ \vec{k}_y$

Nutation : θ

Precession : ψ

Intrinsic rotation : φ

Magnetization direction : $\vec{d}(t) = -\sin\theta \sin\psi\ \vec{k}_x + \sin\theta\cos\psi\ \vec{k}_y + \cos\theta\ \vec{k}_z$

Instantaneous vector rotation : $\vec{\omega}(t) = -\dot{Y}(t)/R\ \vec{k}_x + \dot{X}(t)/R\ \vec{k}_y + \omega_z(t)\ \vec{k}_z$

$\Longrightarrow \dot{\vec{d}} = \vec{\omega} \times \vec{d}$

Eddy currents in plate

$\vec{T} - \Omega$ model (strong form) :

$$\text{In } D \begin{cases} \vec{\nabla} \times \left(\frac{1}{\sigma} \vec{\nabla} \times \vec{T} \right) + \mu_0 \, \partial_t \left(\vec{T} + \vec{\nabla}\Omega \right) + \mu_0 \, \dot{\vec{h}}_s = \vec{0} \\ \vec{\nabla} \cdot \vec{T} = \vec{0} \end{cases} ; \quad \begin{array}{l} \text{On } \partial D \; : \vec{T} \times \vec{n} = \vec{0} \\ \text{In } E_3/D : \vec{T} = \vec{0} \end{array}$$

$$\text{In } E_3 \; : \; \vec{\nabla} \cdot \left[\mu_0 \left(\vec{\nabla}\Omega + \vec{T} \right) \right] = 0 \iff \Omega(t, \vec{x}) = \frac{\mu_0}{4\pi} \int_{\partial D} \frac{\vec{T}(t, \vec{y}) \cdot \vec{n}}{|\vec{x} - \vec{y}|} \, d\vec{y}^2$$

Induced currents : $\vec{j} = \vec{\nabla} \times \vec{T}$

$$\Downarrow$$

Force and torque on magnet for the motion of magnet :

$$\vec{f} = -\int_D \left(\vec{\nabla} \times \vec{T} \right) \times \mu_0 \, \vec{h}_s \, d\vec{x}^{\,3} = \int_{\partial D_s} \mu_0 \mathcal{M} (\vec{d} \cdot \vec{\nabla}\Omega) \; \vec{n} \; d\vec{x}^{\,2}$$

$$\vec{\Gamma} = -\int_D (\vec{x} - \vec{X}) \times \left(\left(\vec{\nabla} \times \vec{T} \right) \times \mu_0 \, \vec{h}_s \right) \; d\vec{x}^{\,3} = \int_{D_s} \mu_0 \mathcal{M} (\vec{d} \times \vec{\nabla}\Omega) \; d\vec{x}^{\,3}$$

Computed in the plate — on the magnet

Motion of magnet

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ \dot X \\ \dot Y \\ \omega_z \\ \theta \\ \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \dot X \\ \dot Y \\ \frac{5}{7m}(f_x + \frac{\Gamma_y}{R}) \\ \frac{5}{7m}(f_y - \frac{\Gamma_x}{R}) \\ \frac{5}{2mR^2}\Gamma_z \\ \frac{1}{R}(\dot X \sin\psi - \dot Y \cos\psi) \\ \omega_z + \frac{1}{R\tan\theta}(\dot X \cos\psi + \dot Y \sin\psi) \\ -\frac{1}{R\sin\theta}(\dot X \cos\psi + \dot Y \sin\psi) \end{pmatrix}$$

where $\begin{cases} \vec f = f_x\,\vec k_x + f_y\,\vec k_y + \not{f_z}\,\vec k_z \\ \qquad f_z > 0 \text{ but } f_z < m\,g \\ \vec\Gamma = \Gamma_x\,\vec k_x + \Gamma_y\,\vec k_y + \Gamma_z\,\vec k_z \end{cases}$

are computed with the $\vec T - \Omega$ model

No viscous or solid frictions are considered

The single damping term is due to the eddy currents and contained in $\vec f, \vec\Gamma$

$$\Downarrow$$

Source magnetic field for the $\vec T - \Omega$ model

$$\vec h_s = \vec\nabla_{\vec x}\,(\Omega_s) \quad \text{with} \quad \Omega_s(t,\vec x) = -\frac{\mathcal{M}\,\vec d \cdot \overrightarrow{xX}}{4\pi\,|\overrightarrow{xX}|^3} \quad \text{and} \quad \overrightarrow{xX} = \vec x - \vec X$$

$$\dot{\vec h}_s = \vec\nabla_{\vec x}\left(\dot\Omega_s\right) \quad \text{with} \quad \dot\Omega_s = \frac{\mathcal{M}}{4\pi\,|\overrightarrow{xX}|^3}\,\vec d \cdot \left(\dot{\vec X} - 3\,\frac{\dot{\vec X} \cdot \overrightarrow{xX}}{|\overrightarrow{xX}|^2}\,\overrightarrow{xX} - \vec\omega \times \overrightarrow{xX}\right)$$

Numerical models

A time step :

$$\Omega^n, \vec{T}^n, \dot{X}^n, \dot{Y}^n, \omega_z^n, X^n, Y^n, \theta^n, \psi^n, \varphi^n \longrightarrow \Omega^{n+1}, \vec{T}^{n+1}, \dot{X}^{n+1}, \dot{Y}^{n+1}, \omega_z^{n+1}, X^{n+1}, Y^{n+1}, \theta^{n+1}, \psi^{n+1}, \varphi^{n+1}$$

Time $\quad t = n\,\tau \quad\quad t + \tau$

Explicit Runge-Kutta (The 5-order of Dormand-Prince pair) for $\dot{X}, \dot{Y}, \omega_z, X, Y, \theta, \psi, \varphi$ but with $f_x, f_y, \Gamma_x, \Gamma_y, \Gamma_z$ frozen at time t

Euler Implicit for $\vec{T} - \Omega$ (weak form) :

$$\forall\, \vec{T}' \;:\; \int_D \left(\frac{1}{\sigma} \vec{\nabla} \times \vec{T}^{n+1} \cdot \vec{\nabla} \times \vec{T}' + \frac{\mu_0}{\tau} \left[\vec{T}^{n+1} + \vec{\nabla} \Omega^{n+1} \right] \cdot \vec{T}' \right) d\vec{x}^{\,3}$$

$$= \int_D \frac{\mu_0}{\tau} \left[\vec{T}^n + \vec{\nabla} \Omega^n \right] \cdot \vec{T}' d\vec{x}^{\,3} - \int_D \mu_0 \, \dot{\vec{h}}_s^{*\;n+1} \cdot \vec{T}' d\vec{x}^{\,3}$$

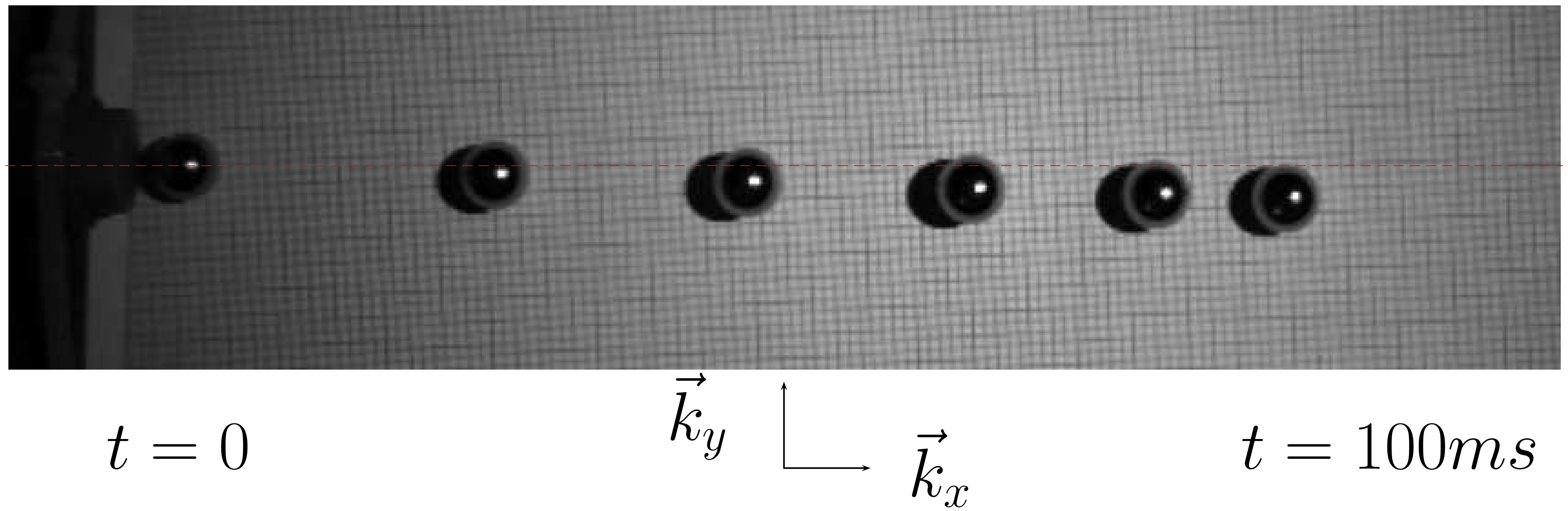
$$\Omega^{n+1}(\vec{x}) = \frac{\mu_0}{4\pi} \int_{\partial D} \frac{\vec{T}^{n+1}(\vec{y}) \cdot \vec{n}}{|\vec{x} - \vec{y}|} d\vec{y}^{\,2}$$

with

$$\dot{\vec{h}}_s^{*\;n} = \vec{\nabla}_{\vec{x}} \left(\frac{\mathcal{M}\; \vec{d}^n}{4\pi\; |\vec{x} - \vec{X}^n|^3} \cdot \left(\dot{\vec{X}}^n - 3\, \frac{\dot{\vec{X}}^n \;\cdot\; \left(\vec{x} - \vec{X}^n\right)}{|\vec{x} - \vec{X}^n|^2} \left(\vec{x} - \vec{X}^n\right) - \vec{\omega}^n \times \left(\vec{x} - \vec{X}^n\right) \right) \right)$$

Comparison with experiment

Experiment : The magnet is set in motion by a compression spring

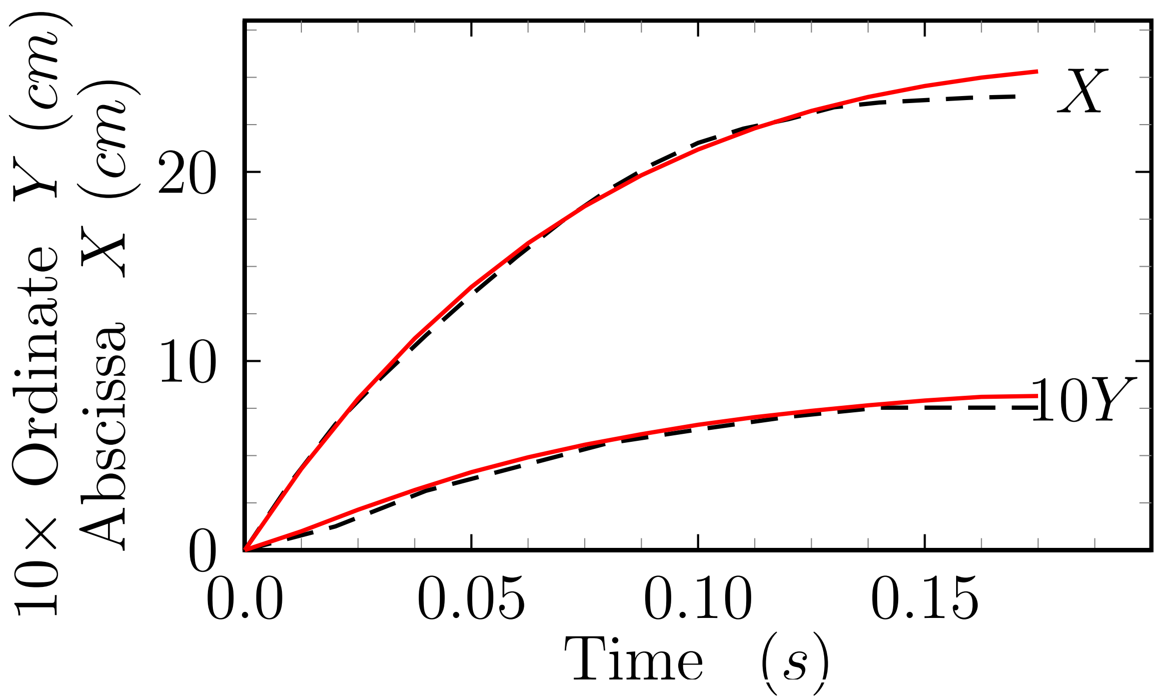


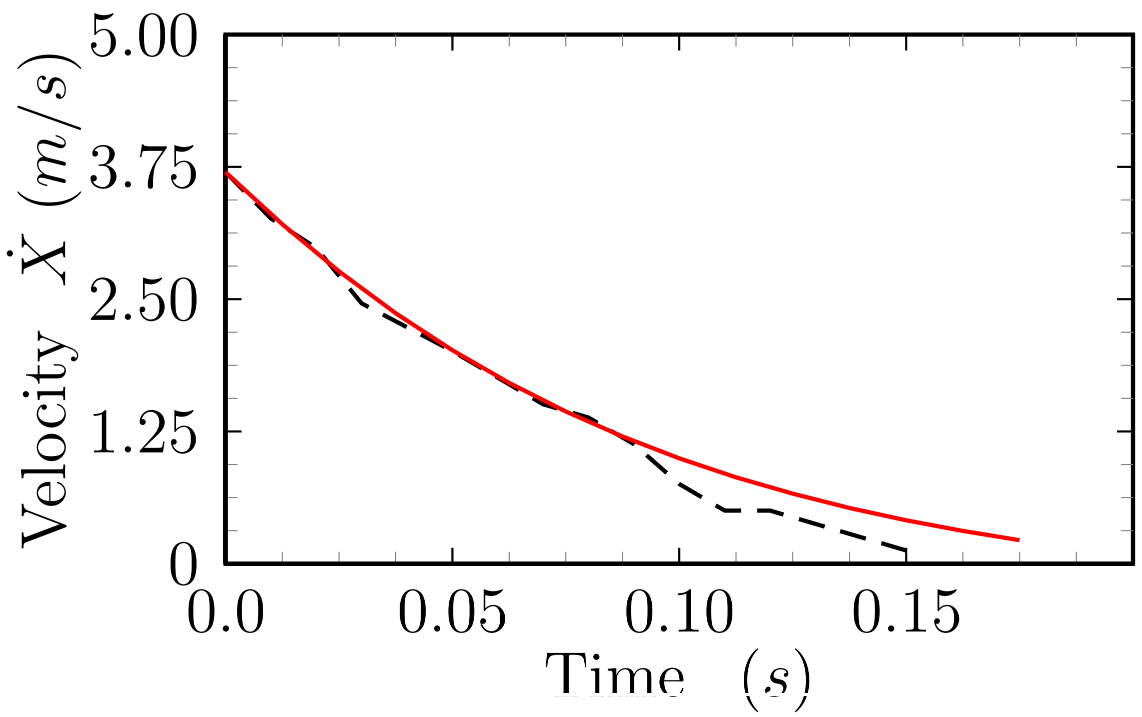
Initial velocity : 3.7m/s (measured)

Initial direction : $\psi_0 = 0^o$; $\theta_0 = 25\,^o$ (direction of terrestrial magnetism)

Trajectory filmed with high-speed camera 500 fps, 6px/mm

Position and velocity vs time : computation : plain/red ; experiment dashed/black

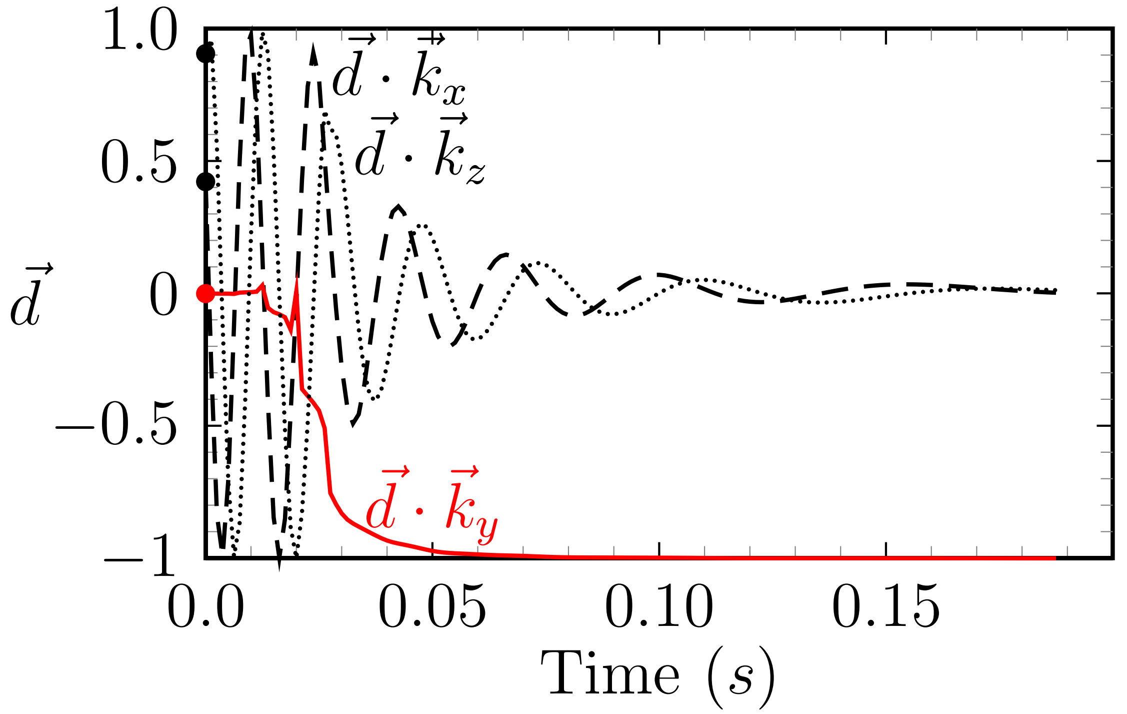


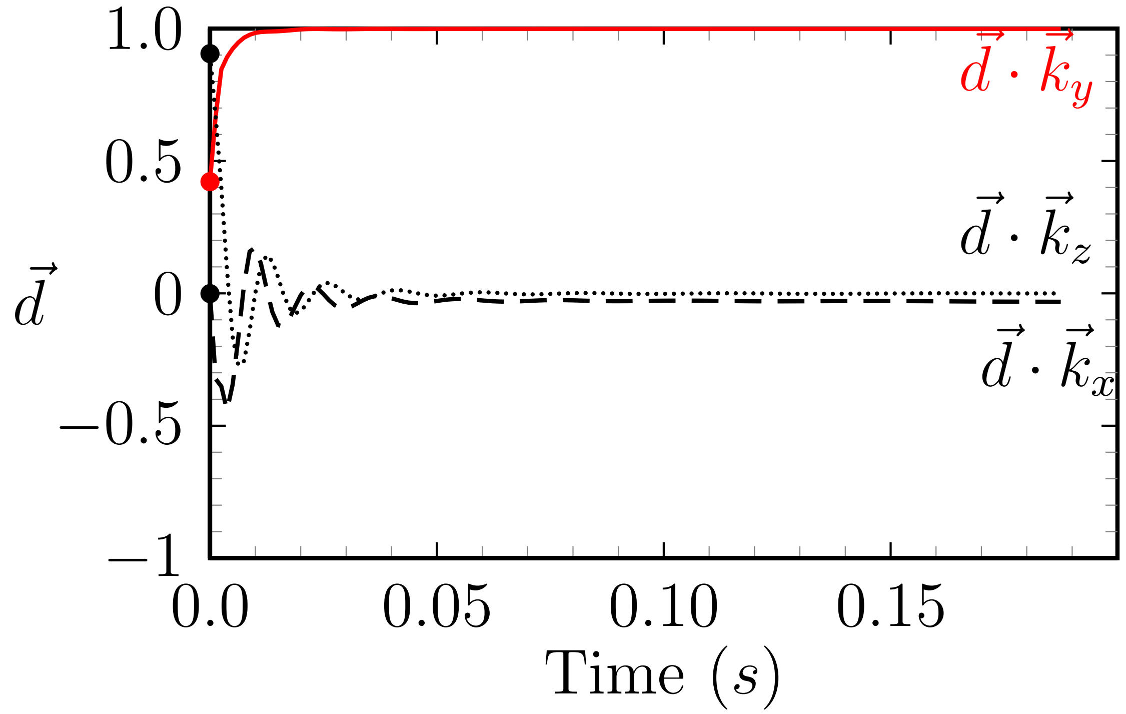


The Y-deviation due to the initial condition on θ_0 is noticed

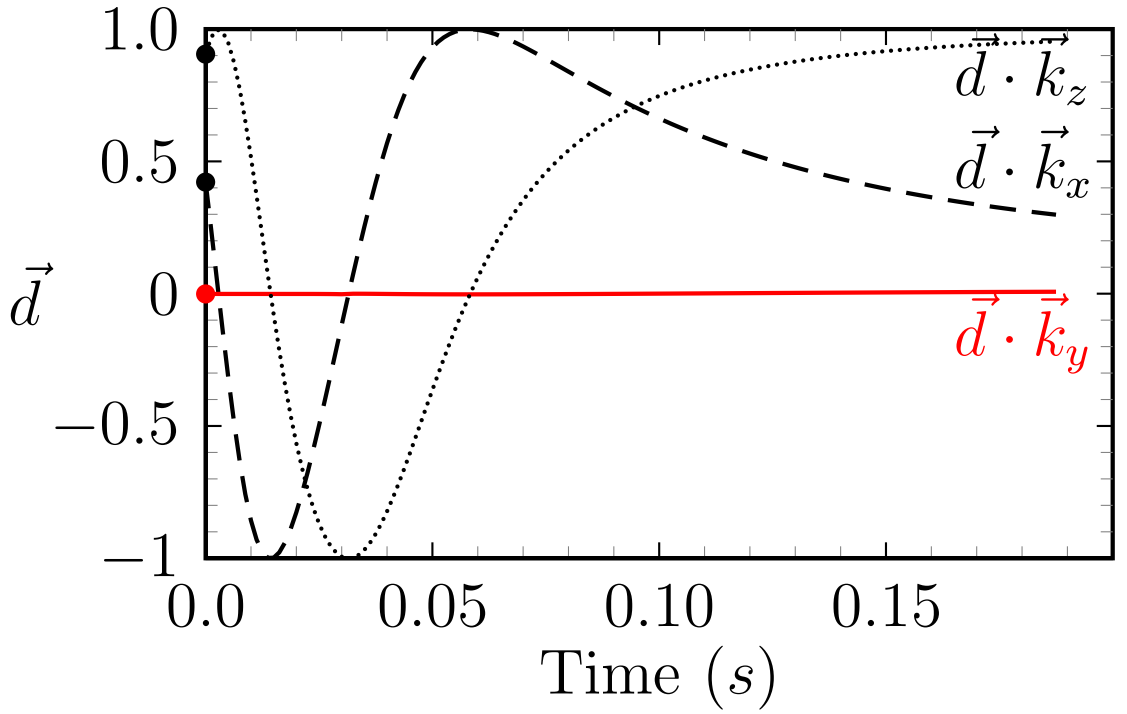
Direction of magnetization

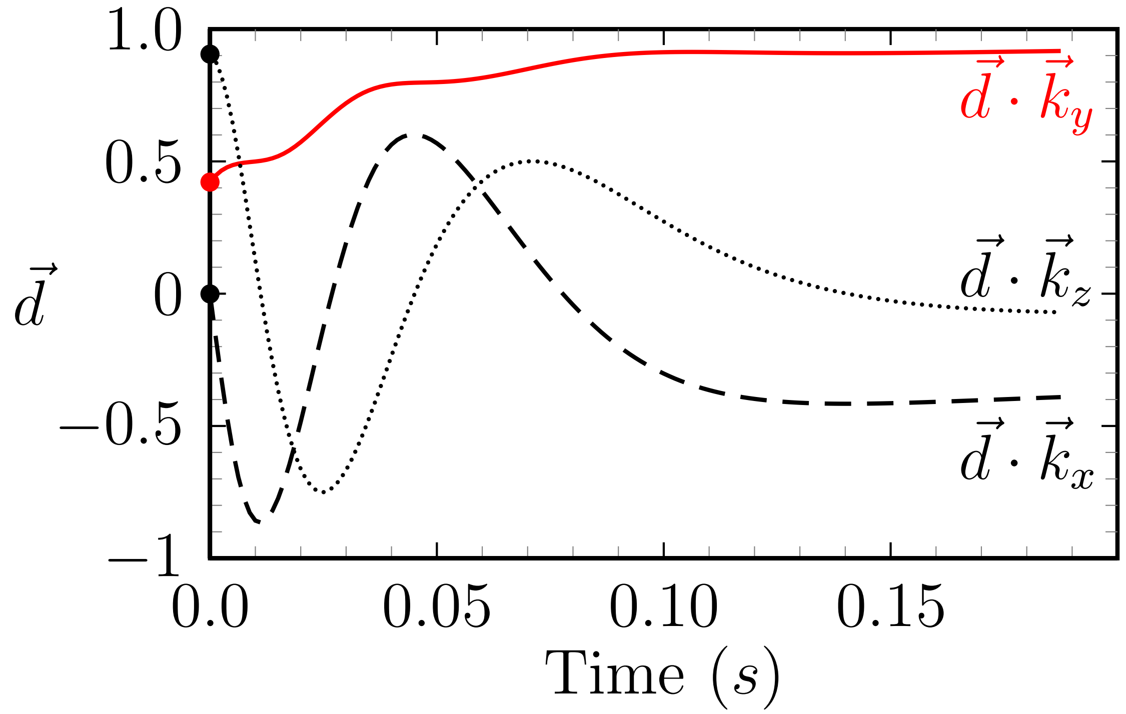
With high initial velocity (3.7m/s), $\vec{d}$ quickly aligns with the transverse axis $\vec{k}_y$





But with low initial velocity (1m/s), the behavior is different





Control of accuracy with energy balance

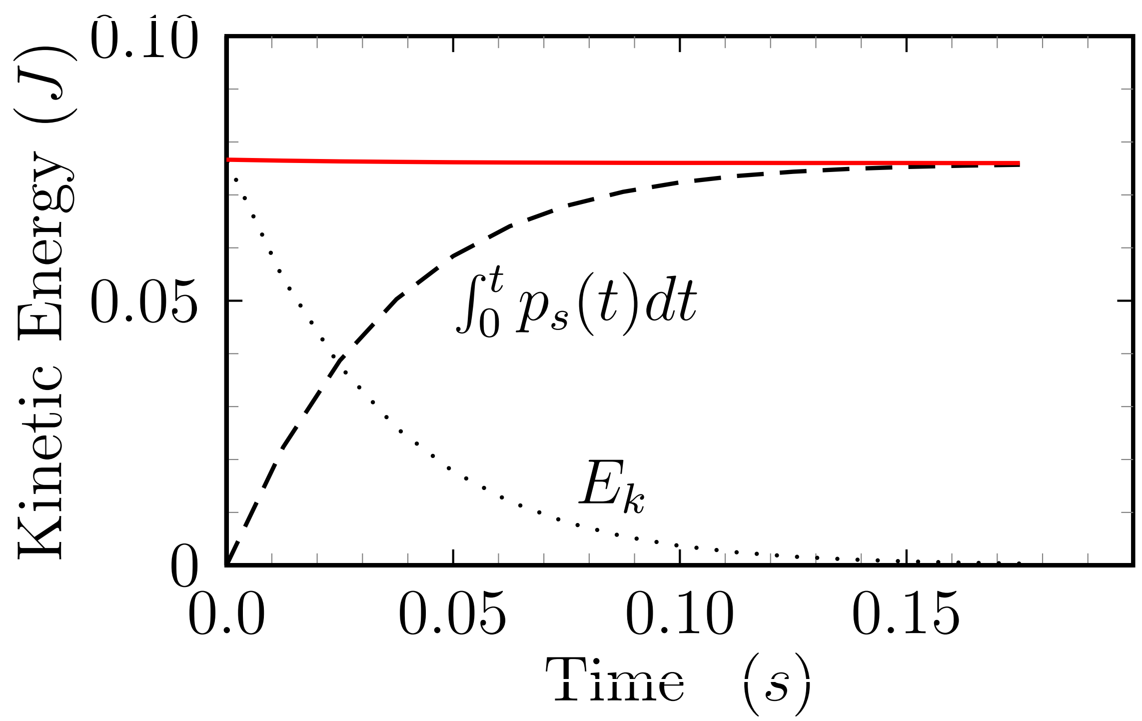
Joule power losses : $p_j = \int_D \frac{\vec{\nabla} \times \vec{T}}{\sigma} \, d\vec{x}^3$

Kinetic energy : $E_k = \frac{1}{2}\left(\frac{7}{5}\, m\, \dot{\vec{X}}^2 + \frac{2}{5}\, m\, R^2\, \omega_z^2\right)$

Magnetic coenergy : $W = \frac{1}{2}\int_{E_3} \mu_0\, \left(\vec{T} + \vec{\nabla}\Omega\right)^2\, d\vec{x}^3$

supplied mechanical power $p_m = \vec{f} \cdot \dot{\vec{X}} + \vec{\Gamma} \cdot \vec{\omega}$

supplied electric power : $p_s = -\int_D \mu_0\, \dot{\vec{h}}_s \cdot \vec{T}\, d\vec{x}^3$



$$\frac{d\overline{W}}{dt} + p_j = p_s(t)\ ;\ \frac{d\overline{W}}{dt} + p_j = p_s(t)$$

$$\Downarrow$$

$$p_s + p_m = 0 \Longrightarrow \frac{d}{dt}\left(E_k + \int_0^t p_s(\tau)\, d\tau\right) = 0$$

The accuracy of the equality allows to control the time step.